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A class of generalised 9-*j* symbols for Sp(2*n*)

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Abstract. The symbolic method of Kramers is used to obtain an expression for a 9-j symbol in terms of multiple products of spinor invariants. This technique is generalised from the unitary (compact) symplectic group Sp(2) to Sp($2n$), and a generating function is found for a class of multiplicity-free 9-$\langle\sigma\rangle$ symbols, where $\langle\sigma\rangle$ denotes an irreducible representation of Sp($2n$) of the form $\langle\sigma_1\sigma_2\ldots\sigma_n\rangle$ for which $\sigma_i=0$ ($i>1$). Schwinger's generating function for a 9-j symbol is recovered by setting $n=1$. A specialisation to 6-$\langle\sigma\rangle$ symbols is made by setting one of the nine irreducible representations equal to the scalar $\langle 00\ldots 0\rangle$, and the method is checked by working out a sample 6-$\langle\sigma\rangle$ symbol previously obtained by an aufbau approach.

1. Kramers' symbolic method

In the early years of quantum mechanics, Kramers (1930) introduced the spinor (ξ, η) when calculating the intensities of certain electronic transitions in atoms. A key ingredient in this approach is the use of the products

$$\xi^{l+m}\eta^{l-m}[(l+m)!(l-m)!]^{-1/2} \qquad (-l\leqslant m\leqslant l) \tag{1}$$

for the $2l+1$ states ϕ_{lm} of a single electron. The factorials in (1) serve to normalise the ϕ_{lm} when they are combined with

$$\partial_\xi^{l+m}\partial_\eta^{l-m}[(l+m)!(l-m)!]^{-1/2} \tag{2}$$

which represent ϕ^*_{lm}. The differential operators $\partial/\partial\xi$ and $\partial/\partial\eta$, which we abbreviate to ∂_ξ and ∂_η, form the spinor (∂_η, $-\partial_\xi$). The angular momentum vector $\boldsymbol{l}$ can be represented by

$$l_+=\xi\partial_\eta \qquad l_-=\eta\partial_\xi \qquad l_z=\tfrac{1}{2}(\xi\partial_\xi-\eta\partial_\eta) \tag{3}$$

where $l_\pm=l_x\pm \mathrm{i}l_y$, and it is easy to verify that the eigenvalues of $\boldsymbol{l}^2$ and l_z for ϕ_{lm} are $l(l+1)$ and m. Instead of ξ, η, ∂_ξ and ∂_η, Schwinger (1965) used the respective creation and annihilation operators $a_+^\dagger$, $a_-^\dagger$, a_+ and a_-, imposing on them the commutation properties of bosons. We prefer to retain the more naive notation of Kramers (1930), partly because the functional roles of the spinors are more transparent and partly to emphasise the connection to the work of such nineteenth century mathematicians as Clebsch and Gordan, as summarised, for example, by Elliott (1895) and Grace and Young (1903). A general description of Kramers' methods has been made by Brinkman (1956).

Unitary transformations of the two components ξ and η form the elements of the group U(2). The parameters (α, β, γ) defining a transformation of the unimodular subgroup SU(2) can be so chosen as to be identical to the Euler angles for a rotation belonging to the group SO(3) whose generators are the components of $\boldsymbol{l}$. In exploiting this correspondence (a homomorphism) between SU(2) and SO(3), Kramers (1930) introduced the notion of a spinor invariant such as $(\xi_1\eta_2 - \eta_1\xi_2)$. Another way of writing this invariant is $\sqrt{2}(\boldsymbol{t}_1^{(1/2)}\boldsymbol{t}_2^{(1/2)})^{(0)}$, where the spinor (ξ_i, η_i) is represented by a spherical tensor $\boldsymbol{t}_i^{(1/2)}$ of rank $\frac{1}{2}$. The notation indicates that the two tensors $\boldsymbol{t}_1^{(1/2)}$ and $\boldsymbol{t}_2^{(1/2)}$ are coupled to a final rank of zero. It is obvious that $(\boldsymbol{t}_i^{(1/2)}\boldsymbol{t}_j^{(1/2)})^{(0)}$ vanishes when $i = j$, so a pair of identical spinors can only be usefully coupled to a rank of 1. In general, a tensorial product $\boldsymbol{t}^{(k)}$ of n identical spinors vanishes unless $k = \frac{1}{2}n$. Put equivalently, successive $\boldsymbol{t}^{(k)}$ can be defined by means of the equation

$$(\boldsymbol{t}^{(k)}\boldsymbol{t}^{(k')})^{(k'')} = \delta(k+k', k'')\boldsymbol{t}^{(k'')} \tag{4}$$

the stretched nature of the coupling preventing any ambiguity in the sequence of construction. The components $(t_q^{(k)})_i$ of a particular $\boldsymbol{t}_i^{(k)}$ are given by

$$\xi_i^{k+q}\eta_i^{k-q}[(k+q)!(k-q)!]^{-1/2} \tag{5}$$

following equation (1). It is easy to confirm that

$$(2k+1)^{1/2}(\boldsymbol{t}_i^{(k)}\boldsymbol{t}_j^{(k)})^{(0)} = (\xi_i\eta_j - \eta_i\xi_j)^{2k}. \tag{6}$$

In a similar way we can define $\boldsymbol{d}^{(1/2)}$ as a tensor with components $(\partial_\eta, -\partial_\xi)$, thereby obtaining

$$(2k+1)^{1/2}(\boldsymbol{d}_i^{(k)}\boldsymbol{d}_j^{(k)})^{(0)} = \Omega_{ij}^{2k} \tag{7}$$

where

$$\Omega_{ij} = \partial^2/\partial\xi_i\partial\eta_j - \partial^2/\partial\eta_i\partial\xi_j. \tag{8}$$

2. Generating function for a 3-*j* symbol

To see how these results can be put to use, consider the product Ξ, given by

$$\Xi = \Theta_{12}^{j_1+j_2-j_3}\Theta_{23}^{j_2+j_3-j_1}\Theta_{31}^{j_3+j_1-j_2} \tag{9}$$

where

$$\Theta_{ij} \equiv \xi_i\eta_j - \eta_i\xi_j. \tag{10}$$

Since Ξ is constructed from invariants Θ_{ij}, it must be an invariant itself. However, on bringing the pairs of tensors $\boldsymbol{t}_i^{(k)}$ with common i together and using equation (4), we eventually arrive at an expression of the form

$$\Xi = \sum_{m_1,m_2,m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \Lambda(j_1j_2j_3)(t_{m_1}^{(j_1)})_1(t_{m_2}^{(j_2)})_2(t_{m_3}^{(j_3)})_3 \tag{11}$$

where the coefficient in large parentheses, the so-called 3-*j* symbol, combines the components of the three tensors to produce an invariant. By equating corresponding powers of ξ_i and η_i in (9) and (11), we can determine the dependence of the 3-*j* symbol on m_1, m_2 and m_3. The calculation is not quite complete, however, since we need to know the factor $\Lambda(j_1j_2j_3)$ in (11). If we take the traditional normalisation

$$\sum_{m_1,m_2,m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^2 = 1 \tag{12}$$

it can be determined by means of the equation

$$\Lambda^2(j_1j_2j_3)=\Omega_{12}^{j_1+j_2-j_3}\Omega_{23}^{j_2+j_3-j_1}\Omega_{31}^{j_3+j_1-j_2}\Theta_{12}^{j_1+j_2-j_3}\Theta_{23}^{j_2+j_3-j_1}\Theta_{31}^{j_3+j_1-j_2}. \tag{13}$$

The right-hand side of this equation is worked out in appendix 1. Taking the square root and including a factor to match the conventional choice of phase, we obtain

$$\Lambda(j_1j_2j_3)=(-1)^{j_1+j_2+j_3}[(j_1+j_2+j_3+1)!(j_1+j_2-j_3)!(j_2+j_3-j_1)!(j_3+j_1-j_2)!]^{1/2}. \tag{14}$$

3. The 9-j symbol

Although our analysis so far may seem somewhat cumbersome for treating 3-j symbols, we can immediately write down an expression for a 9-j symbol by using the well known result that it is equal to a sum over a sextuple product of 3-j symbols (Edmonds 1957, equation (6.4.4)). The only delicate point is to ensure that the pairs of identical m_i values in the sextuple sum are properly matched, and this can be done by introducing the differential operators $\boldsymbol{d}_i^{(k)}$ with ranks k that exactly tally with those of the corresponding $\boldsymbol{t}_i^{(k)}$. Three pairs of equations of the types (9) and (11) are combined with three more pairs in which Θ and t are replaced by Ω and d. Putting the six parts together, we obtain

$$\begin{aligned}
&\Lambda(j_1j_2j_3)\Lambda(j_4j_5j_6)\Lambda(j_7j_8j_9)\Lambda(j_1j_4j_7)\Lambda(j_2j_5j_8)\Lambda(j_3j_6j_9)\\
&\qquad\times\begin{Bmatrix} j_1 & j_2 & j_3\\ j_4 & j_5 & j_6\\ j_7 & j_8 & j_9\end{Bmatrix}\\
&\qquad=\Omega_{14}^{j_1+j_4-j_7}\Omega_{47}^{j_4+j_7-j_1}\Omega_{71}^{j_7+j_1-j_4}\Omega_{25}^{j_2+j_5-j_8}\Omega_{58}^{j_5+j_8-j_2}\Omega_{82}^{j_8+j_2-j_5}\\
&\qquad\times\Omega_{36}^{j_3+j_6-j_9}\Omega_{69}^{j_6+j_9-j_3}\Omega_{93}^{j_9+j_3-j_6}\Theta_{12}^{j_1+j_2-j_3}\Theta_{23}^{j_2+j_3-j_1}\Theta_{31}^{j_3+j_1-j_2}\\
&\qquad\times\Theta_{45}^{j_4+j_5-j_6}\Theta_{56}^{j_5+j_6-j_4}\Theta_{64}^{j_6+j_4-j_5}\Theta_{78}^{j_7+j_8-j_9}\Theta_{89}^{j_8+j_9-j_7}\Theta_{97}^{j_9+j_7-j_8}.
\end{aligned} \tag{15}$$

The total degree of differentiation is $\Sigma_i j_i$, and this is equal to the degree of the polynomial in the various ξ_j and η_k, so the residue is a number. A procedure for evaluating the right-hand side of equation (15) is described in appendix 2. After rearranging the factorial functions, we conclude that

$$\begin{Bmatrix} j_1 & j_2 & j_3\\ j_4 & j_5 & j_6\\ j_7 & j_8 & j_9\end{Bmatrix}$$

is

$$\Delta(j_1j_2j_3)\Delta(j_4j_5j_6)\Delta(j_7j_8j_9)\Delta(j_1j_4j_7)\Delta(j_2j_5j_8)\Delta(j_3j_6j_9)$$

times the coefficient of (A2.3) in $(1-I_4-I_6)^{-2}$, where I_4 and I_6 are given in equations (A2.1) and (A2.2). The functions $\Delta(j_pj_qj_r)$ can be found from equation (A1.7).

4. Generalisations

Our analysis, though possibly appearing rather complex, is in a form that makes generalisation easy. The crucial point to notice is that $\xi_i\eta_j-\eta_i\xi_j$ is a simple example of an antisymmetric bilinear form, and the invariance of such forms is a characteristic feature of symplectic groups. The root figures for the algebras of SU(2), SO(3) and Sp(2) all consist of two oppositely directed vectors, and the three groups are in consequence locally isomorphic to one another. In fact, we could develop all of angular momentum theory for the unitary compact form of Sp(2) rather than for SO(3).

The simplest generalisation of Sp(2) is to Sp(4). As this is followed through it becomes obvious how to make the extension to Sp($2n$). To begin with, the two-component spinor (ξ, η) is replaced by the four-component spinor $(\xi, \zeta, \lambda, \eta)$. The generators $\boldsymbol{l}$ of Sp(2), given in equations (3), go over into the ten generators of Sp(4). As indicated by Racah (1965, equation 76), the two commuting generators of the type H_i are given by

$$H_1 = \xi\partial_\xi - \eta\partial_\eta \qquad H_2 = \zeta\partial_\zeta - \lambda\partial_\lambda. \tag{16}$$

The collection of states

$$\xi^a\zeta^b\lambda^c\eta^d/(a!b!c!d!)^{1/2} \qquad (a+b+c+d=\sigma) \tag{17}$$

for a fixed σ possesses the eigenvalues $a-d$ of H_1 and $b-c$ of H_2. The highest weight of the array $(a-d, b-c)$ is $\langle\sigma, 0\rangle$, and this symbol (simplified by omitting the comma where practicable) denotes the irreducible representation for which the states serve as a basis. Irreducibility is guaranteed because the number of possibilities for a, b, c and d exactly matches the dimension of $\langle\sigma, 0\rangle$, which, from the formula of Flowers (1952), is $\frac{1}{6}(\sigma+1)(\sigma+2)(\sigma+3)$. It is also the dimension of the totally symmetric representation $[\sigma]$ of U(4). This is only to be expected, since the identity of the a factors ξ, the b factors ζ, etc, in (17) ensures that our basis for $\langle\sigma, 0\rangle$ is built from bosons. The existence of irreducible representations $\langle\sigma_1\sigma_2\rangle$ for which $\sigma_2 \neq 0$ indicates that our analysis can only cope with a limited class of representations of Sp(4) and, in general, we are restricted to those irreducible representations of Sp($2n$) of the type $\langle\sigma 0\ldots 0\rangle$.

Within that limitation, we can proceed without difficulty. Our spinor invariant becomes

$$2(\boldsymbol{t}_1^{\langle 10\rangle}\boldsymbol{t}_2^{\langle 10\rangle})^{\langle 00\rangle} = \xi_1\eta_2 - \eta_1\xi_2 + \zeta_1\lambda_2 - \lambda_1\zeta_2 \tag{18}$$

and the analogue of equation (4) is

$$(\boldsymbol{t}^{\langle\sigma 0\rangle}\boldsymbol{t}^{\langle\sigma' 0\rangle})^{\langle\sigma'' 0\rangle} = \delta(\sigma+\sigma', \sigma'')\boldsymbol{t}^{\langle\sigma'' 0\rangle}. \tag{19}$$

The generalisations of Ω_{ij} and Θ_{ij} run

$$\Omega_{ij} = \partial^2/\partial\xi_i\partial\eta_j - \partial^2/\partial\eta_i\partial\xi_j + \partial^2/\partial\zeta_i\partial\lambda_j - \partial^2/\partial\lambda_i\partial\zeta_j \tag{20}$$

and

$$\Theta_{ij} = \xi_i\eta_j - \eta_i\xi_j + \zeta_i\lambda_j - \lambda_i\zeta_j. \tag{21}$$

Analogues of equations (9) and (11) can at once be written down by making the substitution $j_i \to \frac{1}{2}\sigma_i$ and letting m_i stand for the quartet of symbols $(a_ib_ic_id_i)$ for which $a_i+b_i+c_i+d_i=\sigma_i$. The first significant difference concerns the evaluation of $\Lambda(\sigma_1\sigma_2\sigma_3)$. The method of appendix 1 can be followed as before provided we make the replacement

$$\det|a, \partial_\xi, \partial_\eta|_{ijk} \to \det|a, \partial_\xi, \partial_\eta|_{ijk} + \det|a, \partial_\zeta, \partial_\lambda|_{ijk} \tag{22}$$

in the first of the exponential functions of equation (A1.5). The second function must be similarly extended but, because ∂_ξ and ∂_η commute with all functions of the ζ_i and

λ_j, we simply get the product of two independent parts. As a result of this separation, the power -2 in the prefacing factor on the right-hand side of equation (A1.5) becomes -4. We can see how the generalisation to Sp($2n$) goes: the power is $-2n$ and equation (14) becomes

$$\Lambda(\sigma_1\sigma_2\sigma_3) = (-1)^{(\sigma_1+\sigma_2+\sigma_3)/2}[(\tfrac{1}{2}\sigma_1+\tfrac{1}{2}\sigma_2+\tfrac{1}{2}\sigma_3+2n-1)! \\ \times(\tfrac{1}{2}\sigma_1+\tfrac{1}{2}\sigma_2-\tfrac{1}{2}\sigma_3)!(\tfrac{1}{2}\sigma_2+\tfrac{1}{2}\sigma_3-\tfrac{1}{2}\sigma_1)! \\ \times(\tfrac{1}{2}\sigma_3+\tfrac{1}{2}\sigma_1-\tfrac{1}{2}\sigma_2)!/(2n-1)!]^{1/2}. \tag{23}$$

We can also see how to generalise the analysis for the 9-j symbol. The 9-$\langle\sigma\rangle$ symbol for Sp($2n$), namely

$$\begin{Bmatrix} \langle\sigma_1 0\ldots 0\rangle & \langle\sigma_2 0\ldots 0\rangle & \langle\sigma_3 0\ldots 0\rangle \\ \langle\sigma_4 0\ldots 0\rangle & \langle\sigma_5 0\ldots 0\rangle & \langle\sigma_6 0\ldots 0\rangle \\ \langle\sigma_7 0\ldots 0\rangle & \langle\sigma_8 0\ldots 0\rangle & \langle\sigma_9 0\ldots 0\rangle \end{Bmatrix} \tag{24}$$

is

$$\Delta(\sigma_1\sigma_2\sigma_3)\Delta(\sigma_4\sigma_5\sigma_6)\Delta(\sigma_7\sigma_8\sigma_9)\Delta(\sigma_1\sigma_4\sigma_7)\Delta(\sigma_2\sigma_5\sigma_8)\Delta(\sigma_3\sigma_6\sigma_9)$$

times the coefficient of (A2.3), modified by making the substitutions $j_i \to \frac{1}{2}\sigma_i$ everywhere, in

$$(1-I_4-I_6)^{-2n} \tag{25}$$

where, in analogy to equation (A1.7),

$$\Delta(\sigma_i\sigma_j\sigma_k) = [(\tfrac{1}{2}\sigma_i+\tfrac{1}{2}\sigma_j-\tfrac{1}{2}\sigma_k)!(\tfrac{1}{2}\sigma_j+\tfrac{1}{2}\sigma_k-\tfrac{1}{2}\sigma_i)! \\ \times(\tfrac{1}{2}\sigma_k+\tfrac{1}{2}\sigma_i-\tfrac{1}{2}\sigma_j)!(2n-1)!/(\tfrac{1}{2}\sigma_i+\tfrac{1}{2}\sigma_j+\tfrac{1}{2}\sigma_k+2n-1)!]^{1/2}. \tag{26}$$

5. Discussion

Perhaps the most remarkable feature of our result (25) is that it is no more difficult to evaluate the 9-$\langle\sigma\rangle$ symbol (24) than an ordinary 9-j symbol. The same generating function $(1-I_4-I_6)$ appears, albeit with a different inverse power, but only minor adjustments need to be made to the Δ functions. The demonstration by Wu (1972) that Schwinger's generating function $(1-I_4-I_6)^{-2}$ leads to an expression for the 9-j symbol in terms of a sum over six running indices indicates that a similar sextuple sum occurs for the 9-$\langle\sigma\rangle$ symbol.

No multiplicity labels appear in the 9-$\langle\sigma\rangle$ symbol (24). These would be required for the general case in which representations such as $\langle\sigma_{11}\sigma_{12}\ldots\sigma_{1n}\rangle$ appear. However, the basis functions (17) make it clear that a particular irreducible representation $\langle\sigma 0\ldots 0\rangle$ occurs once or not at all in the decomposition of the Kronecker product $\langle\sigma' 0\ldots 0\rangle\times\langle\sigma'' 0\ldots 0\rangle$, so no additional labels are required. By limiting ourselves to representations of the type $\langle\sigma 0\ldots 0\rangle$ we have effectively made the group simply reducible in the sense of Wigner (1965). A glance at the decompositions of the products $\langle\sigma' 0\ldots 0\rangle\times\langle\sigma'' 0\ldots 0\rangle$ into the representations $\langle\sigma 0\ldots 0\rangle$ given by Wybourne (1970, tables D-8–D-15) shows that, if σ' and σ'' are both odd or both even, then σ is even, but if σ' and σ'' possess opposite parity, then σ is odd. This property guarantees that all the arguments of the factorial functions appearing in equations (23) and (26) are integers.

The group Sp(4) is isomorphic to SO(5), every irreducible representation $\langle\sigma_1\sigma_2\rangle$ of Sp(4) corresponding to $(\frac{1}{2}\sigma_1+\frac{1}{2}\sigma_2, \frac{1}{2}\sigma_1-\frac{1}{2}\sigma_2)$ of SO(5). Thus our general result enables us to find the 9-W symbols of the type

$$\begin{Bmatrix} (w_1w_1) & (w_2w_2) & (w_3w_3) \\ (w_4w_4) & (w_5w_5) & (w_6w_6) \\ (w_7w_7) & (w_8w_8) & (w_9w_9) \end{Bmatrix} \tag{27}$$

of SO(5).

6. Reduction to a 6-$\langle\sigma\rangle$ symbol

If we set $\sigma_9=0$ in (24), the 9-$\langle\sigma\rangle$ symbol vanishes unless $\sigma_6=\sigma_3$ and $\sigma_8=\sigma_7$. The powers of a_3, a_6, b_7 and b_8 in the product (A2.3) are zero, and we can therefore remove all terms in I_4 and I_6 that involve any of these four quantities. In this way we find

$$I_4 \to a_1a_2b_6b_9+a_4a_5b_9b_3+a_8a_9b_1b_4+a_9a_7b_2b_5+a_7a_8b_3b_6$$

$$I_6 \to a_2a_4a_9b_2b_4b_9-a_1a_5a_9b_1b_5b_9.$$

When $(1-I_4-I_6)^{-2n}$ is expanded by the multinomial theorem, b_9 appears in products of the type

$$(a_1a_2b_6b_9)^w(a_4a_5b_9b_3)^x(a_2a_4a_9b_2b_4b_9)^y(-a_1a_5a_9b_1b_5b_9)^z$$

i.e. with a total power of $w+x+y+z$. However, the power of a_1 plus the power of a_4 is the same as this (and equal to σ_7), so nothing is gained by picking out the coefficient of b_9. Accordingly, we set $b_9=1$. A similar argument allows us to take $a_9=1$, and we obtain

$$1-I_4-I_6 \to 1-a_1a_2b_6-a_4a_5b_3-a_8b_1b_4-a_7b_2b_5-a_7a_8b_3b_6-a_2a_4b_2b_4+a_1a_5b_1b_5.$$

The asymmetry in the signs can be removed by making the replacements

$$a_1\to -a_1 \qquad b_1\to -b_1 \qquad b_2\to -b_2 \qquad b_3\to -b_3$$

and we now have

$$\begin{aligned} 1-I_4-I_6 \to\ & 1+a_1a_2b_6+a_4a_5b_3+a_8b_1b_4+a_7b_2b_5+a_7a_8b_3b_6 \\ & +a_2a_4b_2b_4+a_1a_5b_1b_5 \end{aligned} \tag{28}$$

while the phase change, from (A2.3), is $(-1)^\tau$, where

$$\begin{aligned} \tau &= \tfrac{1}{2}(\sigma_4+\sigma_7-\sigma_1)+\tfrac{1}{2}(\sigma_2+\sigma_3-\sigma_1)+\tfrac{1}{2}(\sigma_3+\sigma_1-\sigma_2)+\tfrac{1}{2}(\sigma_1+\sigma_2-\sigma_3) \\ &= \tfrac{1}{2}(\sigma_2+\sigma_3+\sigma_4+\sigma_7). \end{aligned} \tag{29}$$

It can also be shown from equation (26) that

$$\Delta(\sigma_i\sigma_j0)=\delta(\sigma_i,\sigma_j)[\sigma_i!(2n-1)!/(\sigma_i+2n-1)!]^{1/2}=\delta(\sigma_i,\sigma_j)[D\langle\sigma_i0\ldots0\rangle]^{-1/2}$$

where $D\langle\sigma_i0\ldots0\rangle$ is the dimension of $\langle\sigma_i0\ldots0\rangle$. Thus, on putting $\sigma_9=0$, the 9-$\langle\sigma\rangle$ symbol (24) becomes

$$\begin{aligned} &(-1)^\tau\delta(\sigma_3,\sigma_6)\delta(\sigma_7,\sigma_8)[D\langle\sigma_30\ldots0\rangle D\langle\sigma_70\ldots0\rangle]^{-1/2} \\ &\qquad\times\begin{Bmatrix} \langle\sigma_10\ldots0\rangle & \langle\sigma_20\ldots0\rangle & \langle\sigma_30\ldots0\rangle \\ \langle\sigma_50\ldots0\rangle & \langle\sigma_40\ldots0\rangle & \langle\sigma_70\ldots0\rangle \end{Bmatrix} \end{aligned} \tag{30}$$

in which the 6-$\langle\sigma\rangle$ symbol is

$$\Delta(\sigma_1\sigma_2\sigma_3)\Delta(\sigma_2\sigma_5\sigma_7)\Delta(\sigma_1\sigma_4\sigma_7)\Delta(\sigma_3\sigma_4\sigma_5)$$

times the coefficient of

$$\begin{aligned} &a_7^{(\sigma_1+\sigma_4-\sigma_7)/2}a_1^{(\sigma_4+\sigma_7-\sigma_1)/2}a_4^{(\sigma_7+\sigma_1-\sigma_4)/2}a_8^{(\sigma_2+\sigma_5-\sigma_7)/2}\\ &\qquad\times a_2^{(\sigma_5+\sigma_7-\sigma_2)/2}a_5^{(\sigma_7+\sigma_2-\sigma_5)/2}b_3^{(\sigma_1+\sigma_2-\sigma_3)/2}b_1^{(\sigma_2+\sigma_3-\sigma_1)/2}\\ &\qquad\times b_2^{(\sigma_3+\sigma_1-\sigma_2)/2}b_6^{(\sigma_4+\sigma_5-\sigma_3)/2}b_4^{(\sigma_5+\sigma_3-\sigma_4)/2}b_5^{(\sigma_3+\sigma_4-\sigma_5)/2} \end{aligned} \tag{31}$$

in $(1-I_4-I_6)^{-2n}$, where $1-I_4-I_6$ is given in equation (28). The reduction of (24) to (30) exactly matches the collapse of a 9-j symbol to a 6-j symbol when one of the arguments of the former is set equal to zero. The phase exponent τ, given in equation (29), becomes simply $j_2+j_3+j_4+j_7$, in agreement with equation (6.4.14) of Edmonds (1957).

7. An example for Sp(2n)

In the process of developing formulae for various 6-$\langle\sigma\rangle$ symbols for Sp($2n$), Suskin (1986) has derived expressions for several of the type appearing in equation (30). His techniques are somewhat similar to those worked out earlier for SO(n) and G_2 (Judd *et al* 1986, Judd 1986), and are distinct from the methods used here.

As an example of our formulae, we calculate

$$\begin{Bmatrix} \langle\sigma 0\ldots 0\rangle & \langle 10\ldots 0\rangle & \langle\sigma-1,0\ldots 0\rangle \\ \langle 10\ldots 0\rangle & \langle\sigma 0\ldots 0\rangle & \langle 20\ldots 0\rangle \end{Bmatrix}. \tag{32}$$

With the aid of equation (26), we find

$$\begin{aligned} &\Delta(\sigma,1,\sigma-1)=[(\sigma-1)!(2n-1)!/(\sigma+2n-1)!]^{1/2}\\ &\Delta(\sigma,\sigma,2)=[(\sigma-1)!(2n-1)!/(\sigma+2n)!]^{1/2}\\ &\Delta(1,1,2)=[1/2n(2n+1)]^{1/2}. \end{aligned} \tag{33}$$

According to (31), we seek the coefficient of

$$a_7^{\sigma-1}a_1a_4a_2a_5b_3b_2^{\sigma-1}b_6b_5^{\sigma-1} \tag{34}$$

in $(1-I_4-I_6)^{-2n}$. The absence of a_8 and b_4 in (34) allows us to simplify (28) by dropping three terms. We thus arrive at

$$\begin{aligned} &(1+a_1a_2b_6+a_4a_5b_3+a_7b_2b_5+a_1a_5b_1b_5)^{-2n}\\ &\qquad=\sum_{p,q,r,s}(-1)^{p+q+r+s}(2n+p+q+r+s-1)!(a_1a_2b_6)^p(a_4a_5b_3)^q\\ &\qquad\quad\times(a_7b_2b_5)^r(a_1a_5b_1b_5)^s/(2n-1)!p!q!r!s!. \end{aligned} \tag{35}$$

Equating corresponding powers in (34) and (35), we get $p=1$, $q=1$, $r=\sigma-1$, $s=0$, together with five checks on these equations. The coefficient of (34) in (35) is thus

$$(-1)^{\sigma+1}(\sigma+2n)!/(2n-1)!(\sigma-1)!$$

and, on multiplying this by the four Δ functions given in equations (33) (the first being used twice), we obtain

$$(-1)^{\sigma+1}[(\sigma+2n)/2(\sigma+2n-1)]^{1/2}[D\langle\sigma-1,0\ldots0\rangle D\langle 20\ldots0\rangle]^{-1/2}$$

in agreement with Suskin.

8. Concluding remarks

When $n>1$, we can no longer check our results by using the analogues of the sum rules for the 6-j and 9-j symbols. This is because the dummy representations $\langle\sigma_1\sigma_2\ldots\sigma_n\rangle$ over which the sums must be performed will almost always involve some for which $\sigma_i\neq 0$ $(i>1)$, and our techniques cannot be used to calculate them. On the other hand, we can of course use the sum rules, including such elaborations as the Biedenharn-Elliott identity (Edmonds 1957, equation (6.2.12)), to gain information about the 6-$\langle\sigma\rangle$ and 9-$\langle\sigma\rangle$ symbols outside the range of our formulae. Butler and Wybourne (1976) have described how the various equations satisfied by generalised 6-j symbols can be used recursively to compute them, and this method would be alleviated for Sp($2n$) by the values provided by our explicit expressions. At the same time, the procedures of Butler and Wybourne (1976) for choosing phases would need to be re-examined if conflicts in signs are to be avoided. At the very least, however, our results would give checks on magnitudes.

The symmetry properties of the 6-j and 9-j symbols stemming from the symmetries of their generating functions carry over to the 6-$\langle\sigma\rangle$ and 9-$\langle\sigma\rangle$ symbols of the type that we have been investigating. In addition to the obvious extensions involving interchanges of rows or columns, the symmetries discovered by Regge (1959) reappear.

Our methods can be generalised to evaluate other n-$\langle\sigma\rangle$ symbols. Equation (15) is exceptional in that it employs only those spinor invariants of the type Θ_{ij} and Ω_{ij}. In general, the mixed form Φ_{ij}, given by

$$\Phi_{ij}=\xi_i\partial/\partial\eta_j+\eta_i\partial/\partial\xi_j$$

is sometimes needed. This is the so-called polarising operator of Grace and Young (1903). If we had used annihilation and creation operators to express our spinor invariants, Φ_{ij} would be recognised as an operator of the type $(\boldsymbol{a}_i^\dagger\boldsymbol{a}_j)^{(0)}$ that preserves the number of particles. However, very little is gained by a notation that explicitly shows scalar products, since all of our work is based on them.

When the generating function is used to derive an explicit formula for the 6-$\langle\sigma\rangle$ symbol appearing in (30), it is found that there is a close parallelism to the classic formula for a 6-j symbol. A single running index z is required as before but, in addition to changing the Δ functions according to (26) and using $\frac{1}{2}\sigma_i$ instead of j_i, the factorial $(z+1)!$ appearing in the numerator of equation (6.3.7) of Edmonds (1957) must be replaced by $(z+2n-1)!/(2n-1)!$. That is,

$$\begin{aligned}&\begin{Bmatrix}\langle\sigma_1 0\ldots0\rangle & \langle\sigma_2 0\ldots0\rangle & \langle\sigma_3 0\ldots0\rangle\\ \langle\sigma_5 0\ldots0\rangle & \langle\sigma_4 0\ldots0\rangle & \langle\sigma_7 0\ldots0\rangle\end{Bmatrix}\\ &\quad=\Delta(\sigma_1\sigma_2\sigma_3)\Delta(\sigma_2\sigma_5\sigma_7)\Delta(\sigma_1\sigma_4\sigma_7)\Delta(\sigma_3\sigma_4\sigma_5)\\ &\quad\times\sum_z\{(-1)^z(z+2n-1)![(2n-1)!]^{-1}\}\{[z-\tfrac{1}{2}(\sigma_1+\sigma_2+\sigma_3)]!\\ &\quad\times[z-\tfrac{1}{2}(\sigma_2+\sigma_5+\sigma_7)]![z-\tfrac{1}{2}(\sigma_1+\sigma_4+\sigma_7)]![z-\tfrac{1}{2}(\sigma_3+\sigma_4+\sigma_5)]!\\ &\quad\times[\tfrac{1}{2}(\sigma_1+\sigma_2+\sigma_4+\sigma_5)-z]![\tfrac{1}{2}(\sigma_1+\sigma_3+\sigma_5+\sigma_7)-z]!\\ &\quad\times[\tfrac{1}{2}(\sigma_2+\sigma_3+\sigma_4+\sigma_7)-z]!\}^{-1}.\end{aligned}\tag{36}$$

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Appendix 1. Calculation of $\Lambda(j_1 j_2 j_3)$

In order to cope with powers of the differential operators Ω_{ij} acting on products of the Θ_{pq}, we consider first the six spinors

$$(u_1, v_1) \equiv (\partial_{\eta_1}, -\partial_{\xi_1}) \qquad (u_1', v_1') \equiv (\xi_1, \eta_1)$$
$$(u_2, v_2) \equiv (\partial_{\eta_2}, -\partial_{\xi_2}) \qquad (u_2', v_2') \equiv (\xi_2, \eta_2)$$
$$(u_3, v_3) \equiv (\mu, \nu) \qquad (u_3', v_3') \equiv (\mu', \nu')$$

and the evaluation of

$$T = \exp(\det|a, u, v|_{123}) \exp(\det|b, u', v'|_{123}) \tag{A1.1}$$

in the limits $\xi_1 \to 0$, $\eta_1 \to 0$, $\xi_2 \to 0$ and $\eta_2 \to 0$, where

$$\det|x, y, z|_{hij} \equiv \begin{vmatrix} x_h & y_h & z_h \\ x_i & y_i & z_i \\ x_j & y_j & z_j \end{vmatrix}. \tag{A1.2}$$

For the special case for which $a_1 = a_2 = b_1 = b_2 = 0$, equation (A1.1) becomes

$$\begin{aligned} T_0 &= \exp(a_3\Omega_{12}) \exp(b_3\Theta_{12}) \\ &= \sum_n (a_3 b_3)^n (n!)^{-2} \Omega_{12}^n \Theta_{12}^n \\ &= \sum_n (a_3 b_3)^n (n+1) = (1 - a_3 b_3)^{-2} \end{aligned} \tag{A1.3}$$

the penultimate step following from a lemma of Grace and Young (1903, § 26). To determine the dependence of T on a_1, we use a technique of Schwinger (1965, equation C2). We apply $\partial/\partial a_1$ to T, thereby producing the operator $\nu\partial_{\eta_2} + \mu\partial_{\xi_2}$, which can act on $\det|b, u', v'|_{123}$. The result of doing this yields the factor

$$b_1(\mu\nu' - \nu\mu') + b_3(\nu\xi_1 - \mu\eta_1)$$

which, being set between the two exponentials in (A1.1), has to be extricated by means of the commutation relation

$$[\exp(a_3\Omega_{12}), (\nu\xi_1 - \mu\eta_1)] = a_3(\nu\partial_{\eta_2} + \mu\partial_{\xi_2}).$$

The differential operator on the right is the same one that was produced originally by $\partial/\partial a_1$, and we can thus set up a differential equation of the type $\partial T/\partial a_1 = fT$, with the solution $T = T_0 \exp(f a_1)$. This procedure can be repeated with a_2, b_1 and b_2. The final result is

$$T = (1 - a_3 b_3)^{-2} \exp[(a_1 b_1 + a_2 b_2)(\mu\nu' - \nu\mu')(1 - a_3 b_3)^{-1}] \tag{A1.4}$$

which is equivalent to equation (C7) of Schwinger (1965).

We now generalise equation (A1.4) in a different direction from that of Schwinger. With the aid of the equation

$$\exp[a\partial_x;\,(x+c)]\exp(bx)=(1-a)^{-1}\exp[bx/(1-a)]\exp[abc/(1-a)]$$

where $\exp(p;\,q)=\Sigma_n p^n(n!)^{-1}q^n$, we obtain, after some straightforward but lengthy manipulation, the result

$$\begin{aligned}&\exp(\det|a,\partial_\xi,\partial_\eta|_{ijk})\\&\qquad\times\exp(\det|b,\xi,\eta|_{ijk}+\xi_i\nu_i-\eta_i\mu_i+\xi_j\nu_j-\eta_j\mu_j+\xi_k\nu_k-\eta_k\mu_k)\\&\qquad=(1-a_ib_i-a_jb_j-a_kb_k)^{-2}\exp[\det|a,\mu,\nu|_{ijk}\\&\qquad\times(1-a_ib_i-a_jb_j-a_kb_k)^{-1}].\end{aligned}\tag{A1.5}$$

To find $\Lambda^2(j_1j_2j_3)$, we set all μ_l and ν_l to zero and note that the product on the right-hand side of equation (13) is

$$[(j_1+j_2-j_3)!(j_2+j_3-j_1)!(j_3+j_1-j_2)!]^2\tag{A1.6}$$

times the coefficient of

$$(a_3b_3)^{j_1+j_2-j_3}(a_1b_1)^{j_2+j_3-j_1}(a_2b_2)^{j_3+j_1-j_2}$$

on the left-hand side of equation (A1.5). This coefficient can be immediately evaluated by turning to the right-hand side of equation (A1.5), which fixes it at $[\Delta(j_1j_2j_3)]^{-2}$, where, to use a traditional notation,

$$\Delta(j_1j_2j_3)=[(j_1+j_2-j_3)!(j_2+j_3-j_1)!(j_3+j_1-j_2)!/(j_1+j_2+j_3+1)!]^{1/2}.\tag{A1.7}$$

Thus $\Lambda^2(j_1j_2j_3)$ is (A1.6) times $[\Delta(j_1j_2j_3)]^{-2}$, and the result of equation (14) follows.

Appendix 2. Generating function for a 9-j symbol

To evaluate the right-hand side of equation (15), we apply equation (A1.5) three times. We first take $(ijk)\equiv(147)$, $b_1=0$, $\nu_1=b_3\eta_2-b_2\eta_3$, $\mu_1=b_3\xi_2-b_2\xi_3$, together with the extensions over the three cycles (147), (258) and (369) for b_1, ν_1 and μ_1. We next take $(ijk)\equiv(258)$, noting that, in the reapplication of equation (A1.5), we must make the replacements

$$b_2\to a_1b_6b_9$$

$$\nu_2\to b_1\eta_3+a_4b_3b_8\eta_9-a_7b_3b_5\eta_6$$

$$\mu_2\to b_1\xi_3+a_4b_3b_8\xi_9-a_7b_3b_5\xi_6$$

together with their cyclic extensions. The final step involves taking $(ijk)\equiv(369)$. Although b_3, b_6 and b_9 are rather lengthy, we now have $\nu_l=\mu_l=0$, so the exponential in equation (A1.5) becomes 1. We are left with $(1-I_4-I_6)^{-2}$, where

$$\begin{aligned}I_4=&\;a_2a_3b_4b_7+a_3a_1b_5b_8+a_1a_2b_6b_9+a_5a_6b_7b_1+a_6a_4b_8b_2+a_4a_5b_9b_3\\&+a_8a_9b_1b_4+a_9a_7b_2b_5+a_7a_8b_3b_6\end{aligned}\tag{A2.1}$$

$$\begin{aligned}I_6=&\;a_2a_4a_9b_2b_4b_9+a_5a_7a_3b_5b_7b_3+a_8a_1a_6b_8b_1b_6\\&-a_1a_5a_9b_1b_5b_9-a_4a_8a_3b_4b_8b_3-a_7a_2a_6b_7b_2b_6.\end{aligned}\tag{A2.2}$$

We can now evaluate the right-hand side of equation (15) by picking the coefficient of

$$a_7^{j_1+j_4-j_7}a_1^{j_4+j_7-j_1}a_4^{j_7+j_1-j_4}a_8^{j_2+j_5-j_8}a_2^{j_5+j_8-j_2}a_5^{j_8+j_2-j_5}$$
$$\times a_9^{j_3+j_6-j_9}a_3^{j_6+j_9-j_3}a_6^{j_9+j_3-j_6}b_3^{j_1+j_2-j_3}b_1^{j_2+j_3-j_1}b_2^{j_3+j_1-j_2}$$
$$\times b_6^{j_4+j_5-j_6}b_4^{j_5+j_6-j_4}b_5^{j_6+j_4-j_5}b_9^{j_7+j_8-j_9}b_7^{j_8+j_9-j_7}b_8^{j_9+j_7-j_8} \tag{A2.3}$$

in the expansion of $(1-I_4-I_6)^{-2}$ and multiplying the result by $\Pi(j_p+j_q-j_r)!$, where j_p, j_q and j_r run over the 18 combinations appearing as powers in equation (A2.3). This 18-fold product of factorials occurs when the exponentials on the left-hand side of equation (A1.5) are expanded as products of the Ω_{ij} and the Θ_{kl}. The components I_4 and I_6 in the generating function differ in a few signs from those given by Schwinger (1965, equation (4.37)), but are identical to those of Wu (1972, equation 43), who developed the method of Bargmann (1962) based on Laplacian integrals.

References

Bargmann V 1962 *Rev. Mod. Phys.* **34** 829-45

Brinkman H C 1956 *Applications of Spinor Invariants in Atomic Physics* (Amsterdam: North-Holland)

Butler P H and Wybourne B G 1976 *Int. J. Quantum Chem.* **10** 581-98

Edmonds A R 1957 *Angular Momentum in Quantum Mechanics* (Princeton, NJ: Princeton University Press)

Elliott E B 1895 *An Introduction to the Algebra of Quantics* (Oxford: Clarendon)

Flowers B H 1952 *Proc. R. Soc.* A **212** 248-63

Grace J H and Young A 1903 *The Algebra of Invariants* (Cambridge: Cambridge University Press)

Judd B R 1986 *J. Math. Phys.* **27** 2616-22

Judd B R, Lister G M S and O'Brien M C M 1986 *J. Phys. A: Math. Gen.* **19** 2473-86

Kramers H A 1930 *Proc. K. Ned. Akad. Wet.* **33** 953-8 (reprinted in Kramers H A 1956 *Collected Scientific Papers* (Amsterdam: North-Holland) pp 405-10)

Racah G 1965 *Ergebnisse der Exakten Naturwissenschaften* (*Springer Tracts in Modern Physics* **37**) (Berlin: Springer) pp 28-84

Regge T 1959 *Nuovo Cimento* **11** 116-7

Schwinger J 1965 *Quantum Theory of Angular Momentum* ed L C Biedenharn and H van Dam (New York: Academic) pp 229-79

Suskin M A 1986 private communication

Wigner E P 1965 *Quantum Theory of Angular Momentum* ed L C Biedenharn and H van Dam (New York: Academic) pp 87-133

Wu A C T 1972 *J. Math. Phys.* **13** 84-90

Wybourne B G 1970 *Symmetry Principles and Atomic Spectroscopy* (New York: Wiley-Interscience)